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# An Elementary Approach to Inverse Approximation Theorems

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The purpose of this note is to emphasize the possibility of proving inverse theorems in approximation theory without using Bernstein's telescoping argument (for the latter cf. [8, pp. 99 ff., 109 ff., 146 ff.]). Originally this was motivated by a paper of Berens and Lorentz [7] in which they offered an elementary proof of the inverse theorem for Bernstein polynomials in case the exponent $\alpha$ satisfies $0 < \alpha < 1$. For the other values, $1 \leqslant \alpha < 2$, they had to proceed via some intricate arguments using intermediate space methods, but the hope was expressed that the elementary method might be extended to all values $\alpha$, $0 < \alpha < 2$. This is indeed the case but will be worked out in [1].

In this note we would like to illustrate the method in two typical situations including inverse theorems for families of commutative operators. To be specific, let $C_{2\pi}$ be the space of $2\pi$-periodic continuous functions $f$ with norm $\|f\| := \max |f(x)|$. First, we consider a family of convolution operators

$$T_\rho f(x) := (f * k_\rho)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\, k_\rho(u)\, du \tag{1}$$

with (smooth) kernel $\{k_\rho(x)\}_{\rho>0} \subset L^1_{2\pi}$ depending upon a parameter $\rho > 0$ tending to infinity. In view of the translation invariance one has

$$(T_\rho[f])'(x) = T_\rho[f'](x). \tag{2}$$

It is assumed that the operators are uniformly bounded

$$\| T_\rho f\| \leqslant M \|f\| \tag{3}$$

and satisfy a Bernstein-type inequality

$$\|(T_\rho f)''\| \leqslant M\varphi(\rho)^2 \| T_\rho f\| \tag{4}$$



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for all $f$ with $\varphi(\rho) > 0$ monotonely increasing to infinity such that

$$\sup_{\rho>0} \varphi(\rho+1)/\varphi(\rho) := K < \infty. \tag{5}$$

For $\delta > 0$, $0 < \alpha \leqslant 2$ let

$$\omega_2(f;\delta) := \sup_{0<h\leqslant\delta} \|f(x+h) - 2f(x) + f(x-h)\| := \sup_{0<h\leqslant\delta} \|\Delta_h^2 f(x)\|.$$

$$\mathrm{Lip}_2\, \alpha := \{f \in C_{2\pi};\ \omega_2(f;\delta) = O(\delta^\alpha),\ \delta \to 0+\}.$$

By the monotonicity of the modulus of continuity it follows that

$$w_2(f;h) \leqslant M[t^\alpha + (h/t)^2\, w_2(f;t)] \qquad (h, t > 0) \tag{6}$$

implies $f \in \mathrm{Lip}_2\, \alpha$ provided $0 < \alpha < 2$. Indeed, one has (cf. [7, p. 696])

LEMMA. *Let $\Omega$ be monotonely increasing on* $[0, c]$. *Then* $\Omega(t) = O(t^\alpha)$, $t \to 0+$, *if for some* $0 < \alpha < r$ *and all* $h, t \in [0, c]$

$$\Omega(h) \leqslant M[t^\alpha + (h/t)^r\, \Omega(t)]. \tag{7}$$

*Proof.* Let $\mathbb{N}$ be the set of natural numbers and $A > 1$ be such that $2M \leqslant A^{r-\alpha}$. With

$$M_1 := \max\{c^{-\alpha}\Omega(c), 2MA^\alpha\}, \qquad h_m := cA^{1-m}, \qquad (m \in \mathbb{N}),$$

one has $\Omega(h_m) \leqslant M_1 h_m^\alpha$ via induction. Indeed, $\Omega(h_1) = \Omega(c) \leqslant M_1 h_1^\alpha$, whereas (7) for $h = h_m$, $t = h_{m-1}$ delivers

$$\begin{aligned}\Omega(h_m) &\leqslant M[h_{m-1}^\alpha + (h_m/h_{m-1})^r\, \Omega(h_{m-1})]\\ &\leqslant M[A^\alpha h_m^\alpha + A^{-r} M_1 h_{m-1}^\alpha]\\ &\leqslant [MA^\alpha + MA^{\alpha-r} M_1]\, h_m^\alpha \leqslant M_1 h_m^\alpha.\end{aligned}$$

Let $t \in (0, c]$ be fixed and $m \in \mathbb{N}$ be such that $h_m \leqslant t < h_{m-1}$. Then the monotonicity of $\Omega$ yields

$$\Omega(t) \leqslant \Omega(h_{m-1}) \leqslant M_1 h_{m-1}^\alpha = M_1 A^\alpha h_m^\alpha \leqslant M_1 A^\alpha t^\alpha. \quad ∎$$

Introducing the Steklov means for $\delta > 0$ via

$$f_\delta(x) := \delta^{-2} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} f(x+s+t)\, ds\, dt, \tag{8}$$

it is a well-known fact that (cf. [8, p. 38])

$$\|f - f_\delta\| \leqslant w_2(f; \delta), \qquad \|f''_\delta\| \leqslant \delta^{-2} w_2(f; \delta). \tag{9}$$

In these terms the inverse theorem for the linear, commutative, approximation process (1) reads

THEOREM 1. *If* $0 < \alpha < 2$, *then*

$$\| T_\rho f - f\| \leqslant M\varphi(\rho)^{-\alpha} \Rightarrow f \in \mathrm{Lip}_2\, \alpha.$$

*Proof.* By the assumption and (2 — 4, 9) one has for any $h > 0$

$$\begin{aligned}\| \Delta_h{}^2 f\| &\leqslant \| \Delta_h{}^2(f - T_\rho f)\| + \Big/ \iint_{-h/2}^{h/2} (T_\rho f)''(x + s + t)\, ds\, dt \Big/ \\ &\leqslant 4\, \|f - T_\rho f\| + h^2\{\|(T_\rho[f - f_\delta])''\| + \| T_\rho[f''_\delta]\|\} \\ &\leqslant 4M\varphi(\rho)^{-\alpha} + Mh^2\{\varphi(\rho)^2\, \|f - f_\delta\| + \|f''_\delta\|\} \\ &\leqslant 4M\varphi(\rho)^{-\alpha} + Mh^2\{\varphi(\rho)^2 + \delta^{-2}\}\, \omega_2(f; \delta) \\ &\leqslant M[\delta(\rho)^\alpha + (h/\delta(\rho))^2\, \omega_2(f; \delta(\rho))]\end{aligned}$$

with $\delta = \delta(\rho) := 1/\varphi(\rho)$. By choosing $\rho$ such that $\delta(\rho) \leqslant t < \delta(\rho - 1) \leqslant K\,\delta(\rho)$ (see (5)), this implies (6), thus $f \in \mathrm{Lip}_2\, \alpha$. ▌

Let us turn to the classical Bernstein (Zygmund) theorem concerning the polynomial $t_n{}^*(f)$ of best approximation

$$E_n{}^*(f) := \inf_{t_n \in \Pi_n} \|f - t_n\| = \|f - t_n{}^*(f)\|,$$

$\Pi_n$ being the set of (complex) trigonometric polynomials of degree $n$. Instead of the Steklov means (8) we now use a polynomial process, i.e., let $\{J_n\}_{n \in \mathbb{N}}$ be a sequence of convolution operators (1) satisfying (3) and

$$J_n f \in \Pi_n, \qquad \| J_n f - f\| \leqslant M w_2(f; n^{-1}) \tag{10}$$

for all $f$ and $n$. For example, one may take the Fejér–Korovkin means (cf. [8, p. 80]). By the Bernstein inequality $\| t''_n\| \leqslant n^2 \| t_n\|$ for trigonometric polynomials there follows (cf. (8, 9))

$$\begin{aligned}\|(J_n f)''\| &\leqslant \|(J_n[f - f_{1/n}])''\| + \| J_n[f''_{1/n}]\| \\ &\leqslant n^2 \| J_n[f - f_{1/n}]\| + M\, \|f''_{1/n}\| \leqslant M n^2 w_2(f; n^{-1}).\end{aligned} \tag{11}$$

Obviously, $\| t_n^*(f)\| \leqslant 2 \|f\|$ and $t_n^*(f - J_nf) = t_n^*(f) - J_nf$.

THEOREM 2. *If* $0 < \alpha < 2$, *then*

$$E_n^*(f) = O(n^{-\alpha}) \Rightarrow f \in \mathrm{Lip}_2\, \alpha.$$

*Proof.* Using (10, 11) one may proceed as for Theorem 1 to obtain

$$\begin{aligned}\| \Delta_h^2 f \| &\leqslant \| \Delta_h^2(f - t_n^*(f))\| + \iint_{-h/2}^{h/2} \| t_n^*(f)''(x + s + t)\| \, ds\, dt \\ &\leqslant 4E_n^*(f) + h^2\{\|[t_n^*(f - J_nf)]''\| + \|[J_nf]''\|\} \\ &\leqslant 4Mn^{-\alpha} + (nh)^2\{\| t_n^*(f - J_nf)\| + M\omega_2(f; n^{-1})\} \\ &\leqslant M[n^{-\alpha} + (nh)^2\, \omega_2(f; n^{-1})] = M[\delta_n^\alpha + (h/\delta_n)^2\, \omega_2(f; \delta_n)]\end{aligned}$$

with $\delta_n := 1/n$. Since $\delta_n/\delta_{n+1} \leqslant 2$, this implies (6), thus $f \in \mathrm{Lip}_2\, \alpha$. ∎

The use of suitable regularization processes such as (8) is quite standard in connection with direct approximation theorems. The above shows that their appropriate use also enables one to follow what one may call (cf. the comments in [10, p. 69]) a straightforward approach to *inverse* results. In fact, the proofs of Theorems 1 and 2 only need a Bernstein-type inequality plus the *direct* theorem for a suitable regularization process.

It is almost obvious that the above method works in many other situations, e.g., in further Banach spaces such as $L_{2\pi}^p$ or $L^p(-\infty, \infty)$, $1 \leqslant p < \infty$, for an arbitrary order of approximation (greater than 2), in the study of Zamansky-type results, etc. In fact, one may use this elementary procedure even in those cases where the original telescoping argument seems to fail, for example in the treatment of inverse theorems for noncommutative linear processes such as the Bernstein polynomials (cf. [1, 7]). Further details, however, will be worked out elsewhere (cf. [2–5], see also [6]).

*Note added in proof.* The use of the Steklov means (8) was also employed by Ditzian and May [9] in order to prove an inverse result in the particular situation of *local* approximation by Kantorovitch polynomials, apparently without realizing its methodological applicability in general.

## REFERENCES

1. M. BECKER, An elementary proof of the inverse theorem for Bernstein polynomials, *Aequationes Math.*, to appear.
2. M. BECKER, Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.* **27** (1978), 127–142.

3. M. BECKER, Inverse theorems for Favard operators in polynomial weight spaces, to appear.
4. M. BECKER, D. KUCHARSKI, AND R. J. NESSEL, Global approximation theorems for the Szász-Mirakjan operators in exponential weight spaces, *in*: "Linear Spaces and Approximation" (Proc. Conf. Math. Res. Inst. Oberwolfach, 20.–27.8.1977, Eds. P. L. Butzer and B. Sz.-Nagy), ISNM, Vol. 40, pp. 319–333, Birkhäuser, Basel, 1978.
5. M. BECKER AND R. J. NESSEL, Inverse results via smoothing, *in*: "Constructive Function Theory" (Proc. Intern. Conf., Blagoevgrad, 30.5.–4.6.1977), Sofia, to appear.
6. M. BECKER AND R. J. NESSEL, A global approximation theorem for Meyer-König and Zeller operators, *Math. Z.*, to appear.
7. H. BERENS AND G. G. LORENTZ, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972), 693–708.
8. P. L. BUTZER AND R. J. NESSEL, Fourier Analysis and Approximation, I, Birkhäuser, Basel and Academic Press, New York, 1971.
9. Z. DITZIAN AND C. P. MAY, $L_p$ saturation and inverse theorems for modified Bernstein polynomials, *Indiana Univ. Math. J.* **25** (1976), 733–751.
10. H.S. SHAPIRO, Smoothing and Approximation of Functions, Van Nostrand, Princeton, N.J., 1969.